

Davenport constant for semigroups II

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Abstract

Let \mathcal{S} be a finite commutative semigroup. The Davenport constant of \mathcal{S} , denoted $D(\mathcal{S})$, is defined to be the least positive integer ℓ such that every sequence T of elements in \mathcal{S} of length at least ℓ contains a proper subsequence T' ($T' \neq T$) with the sum of all terms from T' equaling the sum of all terms from T . Let $q > 2$ be a prime power, and let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field \mathbb{F}_q . Let R be a quotient ring of $\mathbb{F}_q[x]$ with $0 \neq R \neq \mathbb{F}_q[x]$. We prove that

$$D(\mathcal{S}_R) = D(U(\mathcal{S}_R)),$$

where \mathcal{S}_R denotes the multiplicative semigroup of the ring R , and $U(\mathcal{S}_R)$ denotes the group of units in \mathcal{S}_R .

Key Words: Davenport constant; Zero-sum; Finite commutative semigroups; Polynomial rings

1 Introduction

Let G be an additive finite abelian group. A sequence T of elements in G is called a *zero-sum sequence* if the sum of all terms of T equals to zero, the identity element of G . The Davenport constant $D(G)$ of G is defined to be the smallest positive integer ℓ such that, every sequence T of elements in G of length at least ℓ contains a nonempty subsequence T' with the sum of all terms of T' equaling zero. Though attributed to H. Davenport who proposed [3] the study of this constant in 1965, K. Rogers [13] had first studied it in 1963 and this reference was somehow missed out by most of the authors in this area. The Davenport constant together with the celebrated Erdős-Ginzburg-Ziv Theorem obtained by P. Erdős, A. Ginzburg and A. Ziv in

1961 were two pioneering researches for Zero-sum Theory (see [7] for a survey) which has been developed into a branch of Combinatorial Number Theory.

Theorem A. [4] (Erdős-Ginzburg-Ziv Theorem) *Every sequence of $2n - 1$ elements in an additive finite abelian group of order n contains a zero-sum subsequence of length n .*

During the past five decades, the Davenport constant and the Erdős-Ginzburg-Ziv Theorem together with a large of related problems have been studied extensively for the setting of groups (see [2, 5, 6, 8, 9, 11, 12] for example). In 2008, the author of this paper and W.D. Gao formulated the definition of the Davenport constant for finite commutative semigroups which is stated as follows.

Definition B. [15] *Let S be a commutative semigroup (not necessary finite). Let T be a sequence of elements in S . We call T reducible if T contains a proper subsequence T' ($T' \neq T$) such that the sum of all terms of T' equals the sum of all terms of T . Define the Davenport constant of the semigroup S , denoted $D(S)$, to be the smallest $\ell \in \mathbb{N} \cup \{\infty\}$ such that every sequence T of length at least ℓ of elements in S is reducible.*

In fact, starting from the research of Factorization Theory in Algebra, A. Geroldinger and F. Halter-Koch in 2006 have formulated another closely related definition, $d(S)$, for any commutative semigroup S , which is called the small Davenport constant. For the completeness, their definition is also stated here.

Definition C. (Definition 2.8.12 in [10]) *For a commutative semigroup S , let $d(S)$ denote the smallest $\ell \in \mathbb{N}_0 \cup \{\infty\}$ with the following property:*

For any $m \in \mathbb{N}$ and $a_1, \dots, a_m \in S$ there exists a subset $I \subset [1, m]$ such that $|I| \leq \ell$ and

$$\sum_{i=1}^m a_i = \sum_{i \in I} a_i.$$

We have the following connection between the (large) Davenport constant $D(S)$ and the small Davenport constant $d(S)$ for any finite commutative semigroup S .

Proposition D. *Let S be a finite commutative semigroup. Then,*

1. $d(S) < \infty$. (See Proposition 2.8.13 in [10].)
2. $D(S) = d(S) + 1$. (See Proposition 1.2 in [1].)

In 2014, the author together with S.D. Adhikari and W.D. Gao [1] also generalized the Erdős-Ginzburg-Ziv Theorem to finite commutative semigroups.

Very recently, H.L. Wang, L.Z. Zhang, Q.H. Wang and Y.K. Qu [16] made a study of the Davenport constant of the multiplicative semigroup of a quotient ring of $\mathbb{F}_p[x]$. Precisely, they proved the following.

Theorem E. *For any prime $p > 2$, let $f(x)$ be a nonconstant polynomial of $\mathbb{F}_p[x]$ such that $f(x)$ factors into a product of pairwise non-associate irreducible polynomials. Let $R = \frac{\mathbb{F}_p[x]}{(f(x))}$. Then*

$$D(S_R) = D(U(S_R)),$$

where S_R denotes the multiplicative semigroup of the quotient ring $\frac{\mathbb{F}_p[x]}{(f(x))}$ and $U(S_R)$ denotes the group of units in S_R .

Moreover, they conjectured that $D(S_R) = D(U(S_R))$ holds true for all prime $p > 2$ and any nonconstant polynomial $f(x) \in \mathbb{F}_p[x]$.

In this paper, we obtained the following result for the quotient ring of the ring of polynomials over any finite field \mathbb{F}_q where $q > 2$. As a special case, we affirmed their conjecture.

Theorem 1.1. *Let $q > 2$ be a prime power, and let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field \mathbb{F}_q . Let R be a quotient ring of $\mathbb{F}_q[x]$ with $0 \neq R \neq \mathbb{F}_q[x]$. Then*

$$D(S_R) = D(U(S_R)),$$

where S_R denotes the multiplicative semigroup of the ring R , and $U(S_R)$ denotes the group of units in S_R .

2 The proof of Theorem 1.1

We begin this section by giving some preliminaries.

Let S be a finite commutative semigroup. The operation on S is denoted by $+$. The identity element of S , denoted 0_S (if exists), is the unique element e of S such that $e + a = a$ for every $a \in S$. If S has an identity element 0_S , let

$$U(S) = \{a \in S : a + a' = 0_S \text{ for some } a' \in S\}$$

be the group of units of \mathcal{S} . For any element $c \in \mathcal{S}$ and any subset $A \subseteq \mathcal{S}$, let

$$\text{St}_A(c) = \{a \in A : a + c = c\}$$

denote the stabilizer of c in A .

On a commutative semigroup \mathcal{S} the Green's preorder, denoted $\leq_{\mathcal{H}}$, is defined by

$$a \leq_{\mathcal{H}} b \Leftrightarrow a = b \text{ or } a = b + c$$

for some $c \in \mathcal{S}$. Green's congruence, denoted \mathcal{H} , is a basic relation introduced by Green for semigroups which is defined by:

$$a \mathcal{H} b \Leftrightarrow a \leq_{\mathcal{H}} b \text{ and } b \leq_{\mathcal{H}} a.$$

For any element a of \mathcal{S} , let H_a be the congruence class by \mathcal{H} containing a . We write $a <_{\mathcal{H}} b$ to mean that $a \leq_{\mathcal{H}} b$ but $H_a \neq H_b$. The following easy fact will be used later.

Lemma 2.1. (*folklore*) *For any element $a \in \mathcal{S}$, $U(\mathcal{S})$ acts on the congruence class H_a and $\text{St}_{U(\mathcal{S})}(a)$ is a subgroup of $U(\mathcal{S})$.*

In what follows, we also need some notations introduced by A. Geroldinger and F. Halter-Koch (see [10]), which are very helpful to dealing with the problems in zero-sum theory and factorization theory.

The sequence T of elements in the semigroups \mathcal{S} is denoted by

$$T = a_1 a_2 \cdot \dots \cdot a_\ell = \prod_{a \in \mathcal{S}} a^{v_a(T)},$$

where $v_a(T)$ denotes the multiplicity of the element a in the sequence T . By \cdot we denote the operation to join sequences. Let T_1, T_2 be two sequences of elements in the semigroups \mathcal{S} . We call T_2 a subsequence of T_1 if

$$v_a(T_2) \leq v_a(T_1)$$

for every element $a \in \mathcal{S}$, denoted by

$$T_2 \mid T_1.$$

In particular, if $T_2 \neq T_1$, we call T_2 a *proper* subsequence of T_1 , and write

$$T_3 = T_1 T_2^{-1}$$

to mean the unique subsequence of T_1 with $T_2 \cdot T_3 = T_1$. Let

$$\sigma(T) = a_1 + a_2 + \dots + a_\ell$$

be the sum of all terms in the sequence T . By λ we denote the empty sequence. If S has an identity element 0_S , we allow $T = \lambda$ and adopt the convention that $\sigma(\lambda) = 0_S$. We say that T is *reducible* if $\sigma(T') = \sigma(T)$ for some proper subsequence T' of T (note that, T' is probably the empty sequence λ if S has the identity element 0_S and $\sigma(T) = 0_S$). Otherwise, we call T *irreducible*. For more related terminology used in additive problems for semigroups, one is referred to [1, 14]. Here, the following two lemmas are necessary.

Lemma 2.2. ([10], Lemma 6.1.3) *Let G be a finite abelian group, and let H be a subgroup of G . Then, $D(G) \geq D(G/H) + D(H) - 1$.*

Lemma 2.3. (see [15], Proposition 1.2) *Let S be a finite commutative semigroup with an identity. Then $D(U(S)) \leq D(S)$.*

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.3, we need only to show that

$$D(\mathcal{S}_R) \leq D(U(\mathcal{S}_R)).$$

Since the ring $\mathbb{F}_q[x]$ is a principal ideal domain and $0 \neq R \neq \mathbb{F}_q[x]$, we have that $R = \mathbb{F}_q[x]/(f)$ for some nonconstant *monic* polynomial $f \in \mathbb{F}_q[x]$. Let

$$f = f_1^{n_1} * f_2^{n_2} * \cdots * f_r^{n_r} \quad (1)$$

be the factorization of $f(x)$ in $\mathbb{F}_q[x]$, where $r \geq 1$, $n_1, n_2, \dots, n_r \geq 1$, and f_1, f_2, \dots, f_r are pairwise non-associate monic irreducible polynomials of $\mathbb{F}_q[x]$. To proceed, we need to introduce some notations.

Take an arbitrary element $a \in \mathcal{S}_R$. Let $\theta_a \in \mathbb{F}_q[x]$ be the unique polynomial corresponding to the element a with the least degree, i.e.,

$$\overline{\theta_a} = \theta_a + (f)$$

is the corresponding form of a in the quotient ring R with

$$\deg(\theta_a) \leq \deg(f) - 1.$$

By $\gcd(\theta_a, f)$ we denote the greatest common divisor of the two polynomials θ_a and f in $\mathbb{F}_q[x]$ (the unique monic polynomial with the greatest degree which divides both θ_a and f), in particular, by (1),

$$\gcd(\theta_a, f) = f_1^{\alpha_1} * f_2^{\alpha_2} * \cdots * f_r^{\alpha_r}$$

where $\alpha_i \in [0, n_i]$ for $i = 1, 2, \dots, r$.

• For notational convenience, we shall write $\text{St}_{U(\mathcal{S}_R)}(\cdot)$ simply as $\text{St}(\cdot)$ in what follows. It is also noteworthy that for any $a, b, c \in \mathcal{S}_R$, $a + b = c$ holds if and only if $\theta_a * \theta_b \equiv \theta_c \pmod{f}$.

Now we prove the following claim.

Claim A. *Let a and b be two elements of \mathcal{S}_R . Then the following conclusions hold:*

- (i) *If $a \leq_{\mathcal{H}} b$ then $\gcd(\theta_b, f) \mid \gcd(\theta_a, f)$ and $\text{St}(b) \subseteq \text{St}(a)$;*
- (ii) *$a \mathcal{H} b \Leftrightarrow \gcd(\theta_b, f) = \gcd(\theta_a, f) \Leftrightarrow \text{St}(b) = \text{St}(a)$.*

Proof of Claim A. Assume $a \leq_{\mathcal{H}} b$. Since \mathcal{S}_R has the identity element $0_{\mathcal{S}_R}$, we have

$$a = b + c \text{ for some } c \in \mathcal{S}_R.$$

It follows that

$$\gcd(\theta_b, f) \mid \gcd(\theta_b * \theta_c, f) = \gcd(\theta_a, f).$$

For any element $d \in \text{St}(b)$, $d + a = d + (b + c) = (d + b) + c = b + c = a$, and so $d \in \text{St}(a)$. It follows that

$$\text{St}(b) \subseteq \text{St}(a).$$

This proves Conclusion (i).

Now we prove Conclusion (ii).

Assume $a \mathcal{H} b$. Then $a \leq_{\mathcal{H}} b$ and $b \leq_{\mathcal{H}} a$. It follows from Conclusion (i) that

$$\gcd(\theta_b, f) = \gcd(\theta_a, f)$$

and

$$\text{St}(b) = \text{St}(a).$$

Assume $\gcd(\theta_b, f) = \gcd(\theta_a, f)$. It follows that there exist polynomials $h, h' \in \mathbb{F}_q[x]$ such that

$$\theta_a * h \equiv \theta_b \pmod{f}$$

and

$$\theta_b * h' \equiv \theta_a \pmod{f}.$$

It follows that $b \leq_{\mathcal{H}} a$ and $a \leq_{\mathcal{H}} b$, i.e.,

$$a \mathcal{H} b.$$

Assume $\text{St}(b) = \text{St}(a)$. To prove $a \mathcal{H} b$, we suppose to the contrary that $a \mathcal{H} b$ does not hold. Then $\gcd(\theta_b, f) \neq \gcd(\theta_a, f)$. We may suppose without loss of generality that there exist integers $k \in [1, r]$ and $m_k \in [1, n_k]$ such that

$$f_k^{m_k} \mid \gcd(\theta_a, f) \quad (2)$$

and

$$f_k^{m_k} \nmid \gcd(\theta_b, f). \quad (3)$$

Let

$$h = \frac{f}{f_k^{m_k}}. \quad (4)$$

Take an element $\xi \in \mathbb{F}_q \setminus \{0_{\mathbb{F}_q}, 1_{\mathbb{F}_q}\}$.

Now we show that

$$\gcd(h + 1_{\mathbb{F}_q}, f) = 1_{\mathbb{F}_q} \quad (5)$$

or

$$\gcd(\xi * h + 1_{\mathbb{F}_q}, f) = 1_{\mathbb{F}_q}. \quad (6)$$

Suppose to the contrary that $\gcd(h + 1_{\mathbb{F}_q}, f) \neq 1_{\mathbb{F}_q}$ and $\gcd(\xi * h + 1_{\mathbb{F}_q}, f) \neq 1_{\mathbb{F}_q}$. By (1) and (4), we have that $f_i \nmid \gcd(h + 1_{\mathbb{F}_q}, f)$ and $f_i \nmid \gcd(\xi * h + 1_{\mathbb{F}_q}, f)$ for each $i \in [1, r] \setminus \{k\}$, and thus $f_k \mid (h + 1_{\mathbb{F}_q})$ and $f_k \mid (\xi * h + 1_{\mathbb{F}_q})$. It follows that $f_k \mid \xi * (h + 1_{\mathbb{F}_q}) - (\xi * h + 1_{\mathbb{F}_q}) = \xi - 1_{\mathbb{F}_q}$, which is absurd. This proves that (5) or (6) holds.

Take an element $d \in \mathcal{S}_R$ with

$$\theta_d \equiv h + 1_{\mathbb{F}_q} \pmod{f}$$

or

$$\theta_d \equiv \xi * h + 1_{\mathbb{F}_q} \pmod{f}$$

according to (5) or (6) holds respectively. It follows that

$$d \in U(\mathcal{S}_R),$$

and follows from (2), (3) and (4) that

$$\theta_a * \theta_d \equiv \theta_a \pmod{f}$$

and

$$\theta_b * \theta_d \not\equiv \theta_b \pmod{f}.$$

That is, $d \in \text{St}(a) \setminus \text{St}(b)$, a contradiction with $\text{St}(a) = \text{St}(b)$. Hence, we have that

$$a \mathcal{H} b.$$

This proves Claim A. □

Let $T = a_1 a_2 \cdot \dots \cdot a_\ell$ be an arbitrary sequence of elements in \mathcal{S}_R of length

$$\ell = D(U(\mathcal{S}_R)).$$

It suffices to show that T contains a proper subsequence T' with $\sigma(T') = \sigma(T)$.

Take a shortest subsequence V of T such that

$$\sigma(V) \mathcal{H} \sigma(T). \tag{7}$$

We may assume without loss of generality that

$$V = a_1 \cdot a_2 \cdot \dots \cdot a_t \quad \text{where } t \in [0, \ell].$$

By the minimality of $|V|$, we derive that

$$0_{\mathcal{S}_R} >_{\mathcal{H}} a_1 >_{\mathcal{H}} (a_1 + a_2) >_{\mathcal{H}} \dots >_{\mathcal{H}} \sum_{i=1}^t a_i.$$

Denote

$$K_0 = \{0_{\mathcal{S}_R}\}$$

and

$$K_i = \text{St}\left(\sum_{j=1}^i a_j\right) \quad \text{for each } i \in [1, t].$$

By Lemma 2.1, K_i is a subgroup of $U(\mathcal{S}_R)$ for each $i \in [1, t]$. Moreover, since $\text{St}(0_{\mathcal{S}_R}) = K_0$, it follows from Claim A that

$$K_0 \leq K_1 \leq K_2 \leq \dots \leq K_t.$$

For $i \in [1, t]$, since $\frac{U(\mathcal{S}_R)}{K_i} \cong \frac{U(\mathcal{S}_R)/K_{i-1}}{K_i/K_{i-1}}$ and $D(K_i/K_{i-1}) \geq 2$, it follows from Lemma 2.2 that

$$\begin{aligned} D(U(\mathcal{S}_R)/K_i) &= D\left(\frac{U(\mathcal{S}_R)/K_{i-1}}{K_i/K_{i-1}}\right) \\ &\leq D(U(\mathcal{S}_R)/K_{i-1}) - (D(K_i/K_{i-1}) - 1) \\ &\leq D(U(\mathcal{S}_R)/K_{i-1}) - 1. \end{aligned}$$

It follows that

$$\begin{aligned}
1 \leq D(U(\mathcal{S}_R)/K_t) &\leq D(U(\mathcal{S}_R)/K_{t-1}) - 1 \\
&\vdots \\
&\leq D(U(\mathcal{S}_R)/K_0) - t \\
&= D(U(\mathcal{S}_R)) - t \\
&= \ell - t \\
&= |TV^{-1}|.
\end{aligned} \tag{8}$$

By (7) and Conclusion (ii) of Claim A, we have

$$\gcd(\theta_{\sigma(V)}, f) = \gcd(\theta_{\sigma(T)}, f). \tag{9}$$

Let

$$\mathcal{J} = \{j \in [1, r] : f_j^{n_j} \mid \theta_{\sigma(T)}\}.$$

By (9), we have that

$$f_i \nmid \theta_a \text{ for each term } a \text{ of } TV^{-1} \text{ and each } i \in [1, r] \setminus \mathcal{J}, \tag{10}$$

and that

$$f_j^{n_j} \mid \theta_{\sigma(V)} \text{ for each } j \in \mathcal{J}. \tag{11}$$

For each term a of TV^{-1} , let \tilde{a} be the element of \mathcal{S}_R such that

$$\theta_{\tilde{a}} \equiv \theta_a \pmod{f_i^{n_i}} \text{ for each } i \in [1, r] \setminus \mathcal{J} \tag{12}$$

and

$$\theta_{\tilde{a}} \equiv 1_{\mathbb{F}_q} \pmod{f_j^{n_j}} \text{ for each } j \in \mathcal{J}. \tag{13}$$

By (10), (12) and (13), we conclude that $\gcd(\theta_{\tilde{a}}, f) = 1_{\mathbb{F}_q}$, i.e.,

$$\tilde{a} \in U(\mathcal{S}_R) \text{ for each term } a \text{ of } TV^{-1}. \tag{14}$$

By (11) and (12), we conclude that

$$\sigma(V) + \tilde{a} = \sigma(V) + a \text{ for each term } a \text{ of } TV^{-1}. \tag{15}$$

By (8) and (14), we have that $\prod_{a \in TV^{-1}} \tilde{a}$ is a nonempty sequence of elements in $U(\mathcal{S}_R)$ of length

$|\prod_{a \in TV^{-1}} \tilde{a}| = |TV^{-1}| \geq D(U(\mathcal{S}_R)/K_t)$. It follows that there exists a **nonempty** subsequence

$$W \mid TV^{-1}$$

such that

$$\sigma\left(\prod_{a|W} \tilde{a}\right) \in K_t$$

which implies

$$\sigma(V) + \sigma\left(\prod_{a|W} \tilde{a}\right) = \sigma(V). \quad (16)$$

By (15) and (16), we conclude that

$$\begin{aligned} \sigma(T) &= \sigma(TW^{-1}V^{-1}) + (\sigma(V) + \sigma(W)) \\ &= \sigma(TW^{-1}V^{-1}) + (\sigma(V) + \sigma(\prod_{a|W} \tilde{a})) \\ &= \sigma(TW^{-1}V^{-1}) + \sigma(V) \\ &= \sigma(TW^{-1}), \end{aligned}$$

and $T' = TW^{-1}$ is the desired proper subsequence of T . This completes the proof of the theorem. \square

3 Concluding remarks

We remark that if R is the quotient ring of $\mathbb{F}_2[x]$, the conclusion $D(S_R) = D(U(S_R))$ does not always hold true. For example, take $f = x * (x + 1) * g \in \mathbb{F}_2[x]$ where $\gcd(x * (x + 1), g) = 1_{\mathbb{F}_2}$. Let $R = \mathbb{F}_2[x]/(f)$. Take a sequence $T = a_1 \cdot a_2 \cdot \dots \cdot a_\ell$, where $\theta_{a_1} = x$, $\theta_{a_2} = x + 1$, and $a_3 \cdot \dots \cdot a_\ell$ is a sequence of elements in $U(S_R)$ of length $\ell - 2 = D(U(S_R)) - 1$ which contains no nonempty subsequence V with $\sigma(V) = 0_{S_R}$. It is easy to verify that T is an irreducible sequence of length $\ell = D(U(S_R)) + 1$, which implies that $D(S_R) \geq \ell + 1 = D(U(S_R)) + 2$. Hence, we close this paper by proposing the following problem.

Problem. Let R be a quotient ring of $\mathbb{F}_2[x]$ with $0 \neq R \neq \mathbb{F}_2[x]$. Determine $D(S_R) - D(U(S_R))$.

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